The pointwise product in Weyl quantization

2004 J. Phys. A: Math. Gen. 37 6693

(http://iopscience.iop.org/0305-4470/37/26/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 6693–6711 PII: S0305-4470(04)74757-5

The pointwise product in Weyl quantization

D A Dubin[1] **and M A Hennings**[2]

[1] Department of Pure Mathematics, The Open University, Milton Keynes MK7 6AA, UK
[2] Sidney Sussex College, Cambridge CB2 3HU, UK

Received 16 January 2004
Published 16 June 2004
Online at stacks.iop.org/JPhysA/37/6693
doi:10.1088/0305-4470/37/26/007

Abstract

We study the $\odot$-product of Bracken [1], which is the Weyl quantized version of the pointwise product of functions in phase space. We prove that it is not compatible with the algebras of finite rank and Hilbert–Schmidt operators. By solving the linearization problem for the special Hermite functions, we are able to express the $\odot$-product in terms of the component operators, mediated by the linearization coefficients. This is applied to finite rank operators and their matrices, and operators whose symbols are radial and angular distributions.

PACS numbers: 03.65.−w, 45.20.−d, 03.65.Sq, 03.65.Ta

1. Introduction

1.1. Quantization

In an effort to understand Schrödinger's formulation of quantum mechanics[3], Weyl constructed a general correspondence between functions on phase space and operators on Hilbert space [3]. The essence of Weyl's construction is the association

$$\mathrm{e}^{\mathrm{i}(ap+bq)} \longrightarrow \mathrm{e}^{\mathrm{i}(aP+bQ)} \tag{1.1a}$$

which we may extend by integration: for 'good enough' functions $F(a,b)$

$$2\pi T(p,q) \mapsto \int_{\mathbb{R}^2} F(a,b)\,\mathrm{e}^{\mathrm{i}(aP+bQ)}\,\mathrm{d}a\,\mathrm{d}b. \tag{1.1b}$$

Here T is the Fourier transform of F, and Q, P are the position and momentum operators, respectively.

This is written for the simplest Schrödinger system: one degree of freedom with no constraints, and this is the only system considered in this paper, though there is no obstruction to considering unconstrained systems with several degrees of freedom.

The phase space of this system, that is, its set of allowed classical momenta and positions, is $\mathbb{R}^2$, but in order to emphasize the physical interpretation, we denote it by Π. Our convention is

[3] We take it for granted that the reader is familiar with the Schrödinger representation on $L^2(\mathbb{R})$.

to take the first coordinate to be the momentum, the second to be the position: $(p, q) \in \Pi$. That this is the correct physical interpretation of the Weyl correspondence is shown by considering $T(p, q) = f(p)$, in which case $f(P)$ results, and similarly for functions of q alone.

The quantization correspondence above comes without a class of functions T being specified, and for purposes of rigour a choice must be made. As we are mapping functions in phase space to operators on Hilbert space, and these operators can well be unbounded, the choice of phase space functions and the choice of smoothness for wavefunctions must be made simultaneously. It turns out that a mathematically convenient choice is to choose the 'functions' in phase space to be tempered distributions $T \in \mathcal{S}'(\Pi)$, the test function space $\mathcal{S}(\mathbb{R})$ as the common domain of the operators, and the space $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ of continuous mappings from this domain into the tempered distributions in $\mathcal{S}'(\mathbb{R})$. Thus we are working in the rigged Hilbert space model for this system, based on the triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}). \tag{1.2}$$

All these spaces are taken to be equipped with their usual topologies, but we will be making use of this only implicitly, through references to continuity. In addition, we refer to elements of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\Pi)$ as test functions, and those of their duals as distributions, although strictly speaking this is an abuse of terminology.

We now outline the version of quantization that follows from these choices[4]. All of our notation and conventions concerning quantization and distributions are taken from our book [3], [DHS] hereafter.

Quantization [2] sets up a bicontinuous linear correspondence between tempered distributions in phase space and continuous linear mappings from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ as generalized quantum observables on the rigged Hilbert space. We write $\mathbf{\Delta}[T]$ for the generalized observable obtained by quantization of the distribution T; conversely, every observable $X \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ is the quantization of a unique distribution $\mathbf{\Delta}^{-1}(X) = T$ in $\mathcal{S}'(\Pi)$, such that $\mathbf{\Delta}[T] = X$. We call T the *symbol* of X and X the *quantization* of T.

Equation (1.1*b*) suffers from the defect that it requires the Fourier transform F of the phase space distribution T which is the physically more immediate quantity. This can be overcome by taking its Fourier transform, at least formally, and then using the result to obtain the mathematically rigorous correspondence we need.

For this we need the Fourier transform of $W(a, b) = \exp \mathrm{i}(aP + bQ)$. This can be obtained by a Bochner integral, but it is simpler to start with the result.

For every $(p, q) \in \Pi$, consider the bounded operator $\mathbf{\Delta}[p, q]$ on $L^2(\mathbb{R})$ given by

$$(\mathbf{\Delta}[p, q]f)(x) = 2\,\mathrm{e}^{2\mathrm{i}p(x-q)} f(2q - x). \tag{1.3}$$

For T regular enough (a test function for example), $\mathbf{\Delta}[T]$ can be given formally by the rule

$$\mathbf{\Delta}[T] = \frac{1}{2\pi} \int_\Pi T(p, q) \mathbf{\Delta}[p, q]\, \mathrm{d}A \tag{1.4}$$

where $\mathrm{d}A = \mathrm{d}p\, \mathrm{d}q$ is the Lebesgue measure on phase space, and this is a Bochner integral. As noted above, rather than justify this directly, we are going to employ a procedure similar to the method of defining unbounded operators by quadratic forms.

Staying with sufficiently regular T for now, substitute (1.4) into the matrix element $\langle \bar{g}, \mathbf{\Delta}[T]f \rangle$. In our rigged Hilbert space formalism, $f, g \in \mathcal{S}(\mathbb{R})$; the complex conjugation on the g is designed to convert certain sesquilinear expressions to complex bilinear ones. Now

[4] As only quantization in the sense of Weyl is considered in this paper, the term quantization is unambiguous. There is no difficulty in considering other schemes, such as P- or Q-ordering, but we will not do so here.

exchange the orders of integration over Π and $\mathbb{R}$ (which we are not attempting to justify). The result is an integral of the form

$$\int_{\Pi} T(p,q)[\mathcal{G}(\bar{g}\otimes f)](p,q)\,\mathrm{d}A$$

where, for fixed $f, g \in \mathcal{S}(\mathbb{R})$, $\mathcal{G}(\bar{g}\otimes f)$ is a test function on phase space.

Examination of $\mathcal{G}(\bar{g}\otimes f)$ shows that $\mathcal{G}$ can be linearly and continuously extended to an invertible and bicontinuous mapping of $\mathcal{S}(\mathbb{R}^2)$ onto $\mathcal{S}(\Pi)$. We emphasize that the domain of $\mathcal{G}$ is a result of our restricting the 'wavefunctions' f and g to be test functions. If, however, it is necessary for f and g to be arbitrary vectors in $L^2(\mathbb{R})$ for some purpose, the formula for $\mathcal{G}$ will extend to that, and $\mathcal{G}(\bar{g}\otimes f)$ will then belong to $L^2(\Pi)$.

We call $\mathcal{G}$ the Wigner transform[5]:

$$\mathcal{G}(F)(p,q) = \frac{1}{2\pi}\int_{\mathbb{R}} F(q+u/2, q-u/2)\,\mathrm{e}^{\mathrm{i}pu}\,\mathrm{d}u \qquad F \in \mathcal{S}(\mathbb{R}^2). \tag{1.5}$$

It is then possible to define $\mathbf{\Delta}[T] \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ for all $T \in \mathcal{S}'(\Pi)$ by

$$[\![\mathbf{\Delta}[T]f, g]\!] = [\![T, \mathcal{G}(g\otimes f)]\!] \qquad f, g \in \mathcal{S}(\mathbb{R}) \tag{1.6}$$

where we are now using the duality pairings[6] between $\mathcal{S}(\mathbb{R})$ and its dual on the left, and between $\mathcal{S}(\Pi)$ and its dual on the right.

1.2. Products

Knowing that every mapping in $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ is the image under $\mathbf{\Delta}$ of a distribution in phase space, can we design a product for distributions, $(S, T) \mapsto S * T$ such that

$$\mathbf{\Delta}[S \star T] = \mathbf{\Delta}[S]\mathbf{\Delta}[T]? \tag{1.7}$$

Subject to the distributions being regular enough, the answer is yes, and such a product was first constructed by von Neumann [13], but now invariably called the Moyal product and attributed to Moyal [5] and in part to Wigner [4]. We shall not need the explicit expression for the Moyal product, but we note that it is not defined for all pairs of distributions. That is to say, if S and T are tempered distributions, the formula for $S * T$ will not yield a tempered distribution in general. The Moyal product has been a successful construction in that it is one of the principal examples of an algebra deformation, and is important in a proper treatment of the classical limit: see Landsman [14] for this topic, including references. It is also a significant technique in the theory of pseudo-differential operators and applications to the theory of partial differential equations (most usually in the variant introduced by Kohn and Nirenberg [15]), see Hörmander for example [16].

Recently, Bracken has suggested considering the reverse problem, imposing a product on operators whose symbol is the pointwise product of distributions [1]. Just as for the Moyal product, it is not possible to define the pointwise product of all pairs of (tempered) distributions. The best known example is that the square of Dirac's delta function cannot be defined in any reasonable way, a result of Schwartz [17]. In consequence, introduction of this new product of operators must be accompanied by a discussion of pointwise multiplication of distributions.

First we must say what we mean by multiplying distributions—when it is possible. The product must have at least these properties: it is commutative, bilinear and for three or more,

[5] This is not a standard name. We trust that there will be no confusion with the notion of the *Wigner function* of a state, which is the symbol of the associated density matrix. We will have no need of Wigner functions here, nor of any quasi-probabilities associated with them.

[6] The symbol $[\![\cdot\,,\cdot]\!]$ is the generic pairing symbol between a topological vector space and its dual.

associative; and if at least one of the factors is a test function, it must reduce to the known product $\mathcal{S}(\Pi) \times \mathcal{S}'(\Pi) \to \mathcal{S}'(\Pi)$,

$$[\![fT, g]\!] = [\![Tf, g]\!] = [\![T, fg]\!] \qquad f, g \in \mathcal{S}(\mathbb{R}^2) \quad T \in \mathcal{S}'(\Pi). \tag{1.8}$$

What is known about extending this product in individual cases is incomplete. In the first place, tempered distributions may be partitioned into two classes, regular and singular. A regular (tempered) distribution is one defined by a polynomially bounded measurable function (which is necessarily locally integrable). If f is such a function, it defines a distribution in the obvious way[7]:

$$[\![f, g]\!] = \int_\Pi f(p,q)g(p,q)\,\mathrm{d}A \qquad g \in \mathcal{S}(\Pi). \tag{1.9}$$

The delta function is not regular in this sense. This is shown, for example, by Gelfand and Shilov [18] (who consider distributions in a wider sense when discussing regularity). Regular distributions have the property that they form an algebra under the pointwise product.

But these are distributions defined by functions. What of singular distributions? The principle that must apply is obvious intuitively, and Vladimirov [19] puts it like this: "To define the product of generalized functions f and g, they must have the following properties: insofar as f is nonregular in the neighbourhood of an arbitrary point, so must g be regular in this neighbourhood. For example, $\delta(x-a)\delta(x-b) = 0$ if $a \neq b$." Microlocal analysis provides at least a sufficient condition for the product of two distributions in $\mathcal{D}'(\mathbb{R}^n)$ with a common singularity, in terms of their wavefront sets, for example Hörmander [16], theorem 8.2.10. However, this condition, or its variants, does not apply to the product of a distribution with itself, nor is it specific to tempered distributions. Since what we want is to identify algebras of tempered distributions under the pointwise product, what is known is insufficient. Bearing in mind Vladimirov's statement, if we are demanding an algebra, then it seems as though it cannot contain any distribution with nonregular points, since its behaviour at such points will not only not be damped by multiplication with itself any (finite) number of times, it will usually be made worse. Although we have not proved this, it follows that such an algebra must contain only regular distributions. If so, the algebra of polynomially bounded measurable functions is the largest such. We intend to consider this problem in detail at a later time. In this paper we will assume only that we may choose pairs of tempered distributions which do have a well-defined product in $\mathcal{S}'(\Pi)$. We call such pairs *multipliable*.

Thus, if $S, T \in \mathcal{S}'(\Pi)$ are multipliable, then for all $f \in \mathcal{S}(\Pi)$,

$$[\![ST, f]\!] = [\![TS, f]\!] \tag{1.10}$$

is well defined. When S and T are regular, there is no problem with setting[8]

$$(ST)(p,q) = S(p,q)T(p,q). \tag{1.11}$$

Equation (1.10) allows us to write Bracken's suggestion as

$$\mathbf{\Delta}[S] \odot \mathbf{\Delta}[T] = \mathbf{\Delta}[ST] \tag{1.12}$$

defining a new distributive, associative and commutative operator product, indicated by the symbol $\odot$.

In order to consider the phase space multiplication ST in a duality pairing directly, we use the Wigner transform to give the following definition:

[7] We are considering distributions on Π, but could equally well consider this problem on $\mathbb{R}^d$, even for $d = 1$.

[8] If it is true that the largest algebra of multipliable distributions is the set of polynomially bounded measurable functions, this equation is true for all pairs in that algebra.

Definition 1. *For any pair S, T, of multipliable distributions, the $\odot$-product of their quantizations is the mapping $\mathbf{\Delta}[S] \odot \mathbf{\Delta}[T] \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ given by*

$$[\![\mathbf{\Delta}[S] \odot \mathbf{\Delta}[T]f, g]\!] = [\![\mathbf{\Delta}[ST]f, g]\!] \tag{1.13a}$$

$$= [\![ST, \mathcal{G}(g \otimes f)]\!] \qquad f, g \in \mathcal{S}(\mathbb{R}). \tag{1.13b}$$

Bracken found the formula for the kernel of $\mathbf{\Delta}[S] \odot \mathbf{\Delta}[T]$ and then considered the role of $\hbar$ as an expansion parameter. One of his concerns was to see, in some sense, how the classical mechanical formalism, transported to Hilbert space by Weyl quantization and the $\odot$-product, is related to the usual operator (quantum mechanical) product, particularly as $\hbar \to 0$.

Our concerns here are different. We wish to investigate how the $\odot$-product conforms to the usual classes of operators. To begin with, we want to consider how the finite rank and Hilbert–Schmidt operators behave under the new product.

Proposition 2. *The class of finite rank operators on $L^2(\mathbb{R})$ is not closed under the $\odot$-product; neither is the class of Hilbert–Schmidt operators.*

Proof. It is standard that an operator on $L^2(\mathbb{R})$ is Hilbert–Schmidt class if and only if its symbol belongs to $L^2(\Pi)$. As $L^2(\Pi)$ is not closed under the pointwise product, the Hilbert–Schmidt operators are not closed under the $\odot$-product.

The proof for the finite rank operators will be given in section 5. □

Given two distributions with compact support, we will show that they can be multiplied together if their supports are disjoint.

Proposition 3. *Let S, T be tempered distributions of compact supports, and suppose their supports are disjoint. Then S and T are multipliable, with $ST = 0$, the zero distribution. Consequently*

$$\mathbf{\Delta}[S] \odot \mathbf{\Delta}[T] = 0. \tag{1.14}$$

Proof. Let $S, T \in \mathcal{S}'(\Pi)$ be distributions of disjoint and compact support. We can find a sequence (T_n) of test functions converging in $\mathcal{S}'(\Pi)$ to T such that the supports of the T_n are nested and decreasing to the support of T and, moreover, none of these supports meets that of S. Then for each n and all $f \in \mathcal{S}(\Pi)$, $T_n f \in \mathcal{S}(\Pi)$ is a test function whose support is no larger than that of T_n. Therefore $[\![S, T_n f]\!] = 0$ for all n, and so

$$\lim_{n\to\infty} [\![ST_n, f]\!] = \lim_{n\to\infty} [\![S, T_n f]\!] = 0$$

for all test functions f. Hence (ST_n) converges to the zero distribution in $\mathcal{S}'(\Pi)$. □

Whatever results hold for other familiar classes of operators (bounded, positive, compact, trace class, ...) will require detailed analysis. The difficulty is that, as with the theory of integral kernels, there are no useful necessary *and* sufficient conditions for distributions T in order that $\mathbf{\Delta}[T]$ be of one of these classes.

Another concern of ours is calculational. By introducing a certain orthonormal basis into the rigged triple, the special Hermite functions described below, a number of properties of the $\odot$-product are made explicit.

2. Marginals, polynomials and the Weyl group

In the language of Weyl quantization theory, *marginals* refers to the quantization of functions of p alone or of q alone.

Simple and well-known considerations tell us that for distributions on phase space depending on p alone, their pointwise and Moyal products, if defined, coincide; similarly for distributions of q alone. Hence, for all integers $m, n \geqslant 0$,

$$P^m \odot P^n = P^m P^n = P^{m+n} \tag{2.1a}$$

$$Q^m \odot Q^n = Q^m Q^n = Q^{m+n}. \tag{2.1b}$$

In particular,

$$I \odot I = I. \tag{2.1c}$$

But the equality of the Moyal and pointwise products, when they both exist, does not hold for distributions depending non-trivially on both p and q, of course. For general polynomials on phase space, the basic formulae were found by Bracken [1], in terms of the so-called Weyl-ordered polynomials. We will content ourselves with giving the $\odot$-product of the Weyl group. The result for any polynomial follows from this by differentiating with respect to the group parameters and linearity. (We note in passing that the set of all polynomials is an algebra of tempered distributions closed under the pointwise product.)

We assume that the properties of the Weyl operators $W(a, b)$ are known. In particular

1. that the canonical commutation rules are
$$W(a, b)W(c, d) = \mathrm{e}^{\mathrm{i}(ad-bc)/2} W(a + c, b + d) \tag{2.2}$$
2. that equation (1.1a) can now be written as $\boldsymbol{\Delta}[E_{(a,b)}] = W(a, b)$, where, for all $(a, b) \in \mathbb{R}^2$, $E_{(a,b)}(p, q) = \mathrm{e}^{\mathrm{i}(ap+bq)}$ and
3. that we may 'integrate' $W(a, b)$ against tempered distributions (the justification is applying the Fourier map to equation (1.6) using the distributional form of the Fourier–Plancherel theorem).

Proposition 4. *For all* $(a, b), (c, d) \in \mathbb{R}^2$,

$$W(a, b) \odot W(c, d) = W(a + c, b + d) \tag{2.3a}$$

and if S and T are distributions whose (ordinary, not twisted) convolution $S * T$ *is a distribution,*

$$W[S] \odot W[T] = 2\pi W[S * T]. \tag{2.3b}$$

Proof. Applying $\boldsymbol{\Delta}$ to the pointwise product $E_{(a,b)}E_{(c,d)} = E_{(a+c,b+d)}$ yields equation (2.3a). Equation (2.3b) then follows by integrating against $S(a, b)T(c, d)$ under the stipulated conditions. □

Of course $S * T$ is well defined precisely when $\mathcal{F}^{-1}(S)$ and $\mathcal{F}^{-1}(T)$ are multipliable.

Comparing equation (2.3a) with the canonical commutation relations, we see that for the $\odot$-product the symplectic phase factor does not appear. If we reintroduce Planck's constant, the argument of that phase factor is $\mathrm{i}\hbar(ad - bc)/2$, which vanishes with $\hbar$. A possible interpretation of this is given by Bracken [1], but it seems to us that this matter needs further consideration, particularly its geometric aspects.

3. Special Hermite functions

As their name implies, the special Hermite functions are related to the Hermite–Gauss functions, otherwise known as the oscillator eigenfunctions. Our conventions for the Hermite–Gauss functions are as follows[9].

First of all we write h_k for the kth Hermite–Gauss function rather than using Dirac's ket notation $|k\rangle$. Our conventions concerning these functions are implicit in their generating function:

$$G_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{\sqrt{2^k k!}} h_k(x) = \pi^{-1/2} \exp(-t^2/4 + xt - x^2/2). \tag{3.1}$$

We will not need any further details here, but in any event we expect that the reader knows these functions very well. What might not be so well-known is that besides being an orthonormal basis for $L^2(\mathbb{R})$, each h_k is a test function. Moreover, the set of Hermite–Gauss functions constitutes a Schauder basis for both $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ (identifying the distribution with the function that defines it). In the unique expansion of test functions with respect to this basis, the coefficient sequences are rapidly decreasing, whilst for distributions they are of polynomial growth.

Associated with the Hermite–Gauss functions are the operators $P_{m,n}$ for all integers $m, n \geqslant 0$:

$$P_{m,n} f = \langle h_n, f\rangle h_m \qquad f \in \mathcal{S}(\mathbb{R}) \quad m, n \geqslant 0 \tag{3.2a}$$

sometimes written as $|m\rangle\langle n|$. Certain properties of the Hermite–Gauss functions can be transferred to the $P_{m,n}$. Orthonormality results in the cancellation law

$$P_{m,n} P_{j,k} = \delta_{n,j} P_{m,k}. \tag{3.2b}$$

Note that the $P_{n,n}$ are orthogonal projection operators. Completeness is equivalent to the decomposition of the identity operator:

$$\sum_{n=0}^{\infty} P_{n,n} = I. \tag{3.2c}$$

Further details concerning the Hermite–Gauss functions can be found in most quantum mechanics textbooks.

We now turn to the special Hermite functions $\{\phi_{m,n} : m, n \geqslant 0\}$. These may be defined through

$$\mathcal{P}_{s,t}(p, q) = 2 \exp\{-p^2 - q^2 + \mathrm{i}s(p - \mathrm{i}q) - \mathrm{i}t(p + \mathrm{i}q) - st/2\} \tag{3.3a}$$

$$= \sum_{m,n=0}^{\infty} \frac{s^m t^n}{\sqrt{2^{m+n} m! n!}} \phi_{m,n}(p, q) \tag{3.3b}$$

from which it follows that, for all integers $m, n \geqslant 0$,

$$\phi_{m,n}(r, \beta) = (-1)^\mu \, \mathrm{i}^{m-n} 2^{1+\delta/2} \sqrt{\frac{\mu!}{M!}} \, \mathrm{e}^{-r^2} r^\delta \, \mathrm{e}^{\mathrm{i}(n-m)\beta} L_\mu^\delta (2r^2). \tag{3.3c}$$

Here and elsewhere, for each pair of positive integers (m, n) we write $\mu = \min\{m, n\}$, $M = \max\{m, n\}$ and $\delta = |m - n| = M - \mu$. In this formula, (r, β) are the polar coordinates in phase space, defined by $p + \mathrm{i}q = r \exp \mathrm{i}\beta$. There is clearly an indeterminacy in the definition

[9] We distinguish the Hermite polynomials as such from the Hermite polynomials multiplied by the Gaussian by calling the latter the Hermite–Gauss functions.

of β, which will be discussed below, but this is of no significance for the above formula. It will be noticed that, in spite of their name, the special Hermite functions are really Laguerre functions.

The special Hermite functions are test functions, $\phi_{m,n} \in \mathcal{S}(\Pi)$, and form an orthogonal, but not normalized, basis for $L^2(\Pi)$:

$$\langle \phi_{m,n}, \phi_{j,k} \rangle = \int_\Pi \overline{\phi_{m,n}(p,q)}\, \phi_{j,k}(p,q)\, \mathrm{d}p\, \mathrm{d}q = 2\pi \delta_{m,j} \delta_{n,k}. \tag{3.4}$$

In close analogy with the Hermite–Gauss functions, they also form a Schauder basis for the test functions $\mathcal{S}(\Pi)$ and for the phase space distributions $\mathcal{S}'(\Pi)$. There is a certain clarity in distinguishing $\phi_{m,n}$ as a test function from its identification as a distribution, which we write $U_{m,n} \in \mathcal{S}'(\Pi)$. It is worth noting that, as this is a distribution defined by a test function, expressions such as $U_{j,k}(p,q) U_{m,n}(p,q)$ are well defined, with the function $U_{j,k} U_{m,n}$ itself a (distribution defined by a) test function. For details of this part of the theory, involving the antilinear embedding of $\mathcal{S}(\Pi)$ densely into $\mathcal{S}'(\Pi)$, see [DHS].

More concretely, the basis property means that if $F \in \mathcal{S}(\Pi)$, we have the expansion

$$F = \sum_{m,n=0}^{\infty} c_{m,n} \phi_{m,n} \qquad c_{m,n} = \langle \phi_{m,n}, F \rangle \tag{3.5a}$$

with the unique expansion sequence $(c_{m,n})$ rapidly decreasing in both indices. Similarly, if $T \in \mathcal{S}'(\Pi)$,

$$T = \sum_{m,n=0}^{\infty} \tau_{m,n} U_{m,n} \qquad \tau_{m,n} = \frac{1}{2\pi} [\![T, \phi_{m,n}]\!] \tag{3.5b}$$

where $(\tau_{m,n})$ is polynomially bounded in both indices. The 2π is a consequence of the lack of normalization, as seen from

$$[\![U_{m,n}, \phi_{j,k}]\!] = 2\pi \delta_{m,j} \delta_{n,k}. \tag{3.5c}$$

Implicit in these formulae is that the expansion for F converges in the Fréchet topology on $\mathcal{S}(\Pi)$ and the expansion for T converges in the strong dual (DF-) topology.

One might wonder where the special Hermite functions come from in the first place. They arise from the identity

$$\phi_{m,n} = 2\pi \mathcal{G}(h_m \otimes h_n). \tag{3.6}$$

It then follows immediately from (3.5c) and (3.6) that

$$\boldsymbol{\Delta}[U_{m,n}] = P_{m,n} \tag{3.7}$$

which is the principal reason for the utility of the special Hermite functions in quantization.

Apply $\boldsymbol{\Delta}$ to (3.5b). As the series converges in the DF topology, it is justified to interchange $\boldsymbol{\Delta}$ and the summation. Using (3.7) then leads to the quantization formula for any tempered distribution T in terms of Hermite–Gauss functions:

$$\boldsymbol{\Delta}[T] = \sum_{m,n=0}^{\infty} \tau_{m,n} P_{m,n}. \tag{3.8}$$

4. The ⊙-product in terms of special Hermite functions

Having introduced the special Hermite basis and seen its intimate connection with Weyl quantization, we now want to use it in connection with the ⊙-product. This requires us to discover how to expand a product ST of distributions in terms of special Hermite functions. We can do this (by linearity and continuity) if we first learn how to expand a product $U_{j,k}U_{m,n}$ as an infinite linear combination of special Hermite functions, which is known in the theory of orthogonal polynomials as the problem of linearization [6].

In detail, we seek the complex numbers $\{m_1, m_2, m_3 \mid n_1, n_2, n_3\}$ satisfying

$$U_{n_1,m_1}U_{n_2,m_2} = \sum_{m_3,n_3=0}^{\infty} \{m_1, m_2, m_3 \mid n_1, n_2, n_3\}\, U_{n_3,m_3}. \tag{4.1}$$

(We shall call these the SH coefficients.) Remembering that the $U_{m,n}$ are test functions, we can write this in terms of function values:

$$\phi_{m_1,n_1}(p,q)\phi_{m_2,n_2}(p,q) = \sum_{m_3,n_3=0}^{\infty} \{m_1, m_2, m_3 \mid n_1, n_2, n_3\}\, \phi_{m_3,n_3}(p,q). \tag{4.2}$$

As $\phi_{m_1,n_1}, \phi_{m_2,n_2} \in \mathcal{S}(\Pi)$, this series is convergent in the topology of $\mathcal{S}(\Pi)$. All we ever use, however, is the weaker condition that it converges in $\mathcal{S}'(\Pi)$.

Using the orthogonality, the SH coefficients are given by the integral expression

$$\{m_1, m_2, m_3 \mid n_1, n_2, n_3\} = \frac{1}{2\pi}\int_{\Pi} \phi_{m_1,n_1}(p,q)\phi_{m_2,n_2}(p,q)\phi_{m_3,n_3}(p,q)\,\mathrm{d}p\,\mathrm{d}q. \tag{4.3}$$

As we were not able to find the solution for the SH coefficients in the literature, we have provided one in the appendix: it may be that this result is of independent interest.

Suppose that we have solved this problem. For any multipliable distributions R, S with expansion coefficient sequences $(\rho_{m,n})$ and $(\sigma_{m,n})$, respectively, we have the DF-convergent expansion

$$RS = \sum_{m,n=0}^{\infty} \tau_{m,n}U_{m,n} \tag{4.4a}$$

with

$$\tau_{m_3,n_3} = \sum_{m_1,\dots,n_2=0}^{\infty} \{m_1, m_2, m_3 \mid n_1, n_2, n_3\}\, \rho_{m_1,n_1}\sigma_{m_2,n_2}. \tag{4.4b}$$

Note that from the multipliability hypothesis it follows that this series converges absolutely and that $(\tau_{m,n})$ is a polynomially bounded sequence.

We can also apply our solution to the ⊙-product, which is why we considered this calculation in the first place.

Since $\boldsymbol{\Delta}[U_{m,n}] = P_{m,n}$ it is now immediate that

$$P_{n_1,m_1} \odot P_{n_2,m_2} = \sum_{m_3,n_3=0}^{\infty} \{m_1, m_2, m_3 \mid n_1, n_2, n_3\}\, P_{n_3,m_3} \tag{4.5a}$$

and, for multipliable distributions R, S (note the index order),

$$\boldsymbol{\Delta}[R] \odot \boldsymbol{\Delta}[S] = \sum_{m_1,\dots,n_3=0}^{\infty} \{m_1, m_2, m_3 \mid n_1, n_2, n_3\}\, \rho_{n_1,m_1}\sigma_{n_2,m_2}P_{n_3,m_3}. \tag{4.5b}$$

This series, which converges in the topology of $\mathcal{S}'(\Pi)$, gives the ⊙-product in terms of quantities that are, in principle, known.

Further details concerning special Hermite functions may be found in [DHS], Folland [11] and Thangavelu [12].

5. Finite rank operators and matrices

By definition, a finite rank operator on $L^2(\mathbb{R})$ is a linear operator whose range is a finite-dimensional (hence closed) linear subspace $\mathcal{M}$. Suppose, in particular, that $\mathcal{M}$ is one dimensional, spanned by the normalized vector f. Then if g is any nonzero vector in $L^2(\mathbb{R})$, the operator $|f\rangle\langle g|$ has range $\mathcal{M}$, and so this gives us the most general form of a rank 1 operator[10]:

$$(|f\rangle\langle g|)h = \langle g, h\rangle f \qquad h \in L^2(\mathbb{R}). \tag{5.1}$$

If the dimension of $\mathcal{M}$ is N, by choosing an orthonormal basis for it, any operator whose range is $\mathcal{M}$ can be written as a sum of N operators of rank 1. In this way, the $\odot$-product of finite rank operators can be reduced to sums of the $\odot$-product of rank 1 operators:

$$|g\rangle\langle f| \odot |u\rangle\langle v|. \tag{5.2}$$

The simplest way to evaluate this (and the only way that we know) is to expand all the functions in terms of Hermite–Gauss functions. For one finite rank operator we have

$$|g\rangle\langle f| = \sum_{m,n=0}^{\infty} \langle h_m, g\rangle\langle f, h_n\rangle P_{m,n}. \tag{5.3}$$

(The hypothesis is that we know f and g, and so, in principle, we know the expansion coefficients which, note, are expressed in terms of inner products and not duality pairings.) Substituting into equation (5.2) yields the general result in terms of known quantities:

Proposition 5. *For $f, g, u, v \in \mathcal{S}(\mathbb{R})$,*

$$|g\rangle\langle f| \odot |v\rangle\langle u| = \sum_{\boldsymbol{m},\boldsymbol{n}} \{\boldsymbol{n} \mid \boldsymbol{m}\} \left[\left\langle h_{m_1}, g\right\rangle\!\left\langle f, h_{n_1}\right\rangle\!\left\langle h_{m_2}, v\right\rangle\!\left\langle u, h_{n_2}\right\rangle\right] P_{m_3,n_3}. \tag{5.4}$$

We have introduced a vector notation for the indices here, with $\boldsymbol{m} = (m_1, m_2, m_3)$, $\boldsymbol{n} = (n_1, n_2, n_3)$, and the sum is over the full range $\{0, 1, 2, \ldots\}$ for each index.

From this result we can deduce the $\odot$-product of any pair of finite rank operators by linearity.

We now know the $\odot$-product of finite rank operators. As we prove in corollary 14 in the appendix, none of the SH coefficients vanish, so it follows that the product of two finite rank operators is no longer of finite rank. Hence this class, the simplest of the standard operator classes, is not closed under the $\odot$-product.

Corollary 6. *The $\odot$-product of any pair of finite rank operators is not of finite rank. In particular, equation (A.12) yields*

$$P_{0,0} \odot P_{0,0} = 4\sum_{m=0}^{\infty} \left(-\frac{1}{3}\right)^{m+1} P_{m,m}. \tag{5.5}$$

Note that this result does not depend on our choice of basis: an operator is either of finite rank or not.

An associated question is whether there is a matrix version of the $\odot$-product, and if so, whether it is a product that is known. (Aside from the usual matrix multiplication, there is the product $(a_{m,n})(b_{m,n}) = (a_{m,n}b_{m,n})$, for instance.) As with all questions relating operators to matrices, the matrix representation depends on the choice of basis in domain and range.

[10] For rank 1 operators the Dirac notation seems the most useful. Otherwise one may use tensor products.

The only result we have in this regard uses the special Hermite basis for both, and is negative insofar as the resulting product seems new.

With these choices of basis, $P_{0,0}$ has the matrix representation

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

and equation (5.5) takes the following matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} = -\frac{4}{3} \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & -\frac{1}{3} & 0 & \dots \\ 0 & 0 & \frac{1}{9} & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \tag{5.6}$$

which is certainly a peculiar result.

For the $\odot$-product of infinite matrices in general we can obtain only partial results in this manner. The reason is that we only know the $\odot$-product in terms of symbols and their quantizations. Using the Hermite–Gauss basis in domain and codomain, the only matrices obtained in this way have rows and columns which are polynomially bounded sequences, and this does not exhaust the set of all infinite matrices.

Conversely, not every infinite matrix represents an operator, even if we can choose the basis at will. As Halmos tells us, it is difficult to tell which matrices do represent operators [8]. A further complication, present even in finite dimensions, is that a given linear transformation is represented by a generally different matrix with every change of basis in the domain and the codomain. (As anyone who has whiled away time calculating matrices of transition will remember.) What we can say is the following.

Proposition 7. *Let R and S be multipliable distributions. Using the special Hermite expansions, the operators $\boldsymbol{\Delta}[R]$ and $\boldsymbol{\Delta}[S]$ are represented by the matrices $(\rho_{m,n})$ and $(\sigma_{m,n})$, with $\rho_{i,j} = (1/2\pi)[\![R, \phi_{i,j}]\!]$ and $\sigma_{i,j} = (1/2\pi)[\![S, \phi_{i,j}]\!]$. The $\odot$-product of these matrices is given by the matrix $(\tau_{m,n})$ in equation (4.4b).*

6. Products for radial quantization

Polar quantization is concerned with the special cases where the phase space distributions depend only on the radius or on the angle. We refer to [DHS] for a technical description of such distributions. Polar coordinates are introduced through $p + \mathrm{i}q = r\exp(\mathrm{i}\beta)$. We will discuss the distributional nature of the angle in the next section; the radius function is $\mathfrak{r}(p,q) = \sqrt{p^2+q^2}$, and the symbol $\mathfrak{r}$ is reserved for this function.

The radial distributions we consider here are of the form

$$\mathfrak{f}_{\mathrm{rad}}(p,q) = \mathfrak{f} \circ \mathfrak{r}(p,q) = \mathfrak{f}\big(\sqrt{p^2+q^2}\big)$$

where $\mathfrak{f}\colon \mathbb{R}^+ \to \mathbb{C}$ is a polynomially bounded continuous function.

The results of quantizing such distributions are as follows:

$$\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{rad}}]h_n = \rho_n(\mathfrak{f})h_n \tag{6.1}$$

where

$$\rho_n(\mathfrak{f}) = \frac{1}{2\pi}\int_{\Pi} \phi_{n,n}(r,\beta)\mathfrak{f}(r)r\,\mathrm{d}r\,\mathrm{d}\beta \tag{6.2}$$

$$= (-1)^n \int_0^{\infty} \mathfrak{f}(\sqrt{u})\,\mathrm{e}^{-u}L_n(2u)\,\mathrm{d}u. \tag{6.3}$$

Hence $\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{rad}}]$ is diagonal: its eigenvectors are the Hermite–Gauss functions and the eigenvalue corresponding to the eigenvector h_n is $\rho_n(\mathfrak{f})$. In terms of the projection operators $P_{n,n}$ we can write

$$\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{rad}}] = \sum_{n=0}^{\infty} \rho_n(\mathfrak{f})P_{n,n}. \tag{6.4}$$

For all $\mathfrak{f}$ of the given class, the series converges in the topology of $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$.

If $\mathfrak{f}, \mathfrak{g}\colon \mathbb{R}^+ \to \mathbb{C}$ are polynomially bounded and continuous, so is $\mathfrak{f}\mathfrak{g}$, the pointwise product in one variable. Consequently the radial distributions we are considering are all pairwise multipliable, with

$$(\mathfrak{f}_{\mathrm{rad}})(\mathfrak{g}_{\mathrm{rad}}) = (\mathfrak{f}\mathfrak{g})_{\mathrm{rad}}. \tag{6.5}$$

The behaviour of the quantizations of the radial distributions under the $\odot$-product is now clear.

Proposition 8. *Let* $\mathfrak{f}, \mathfrak{g} : \mathbb{R}^+ \to \mathbb{C}$ *be continuous and polynomially bounded. Then*

$$\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{rad}}] \odot \boldsymbol{\Delta}[\mathfrak{g}_{\mathrm{rad}}] = \sum_{k=0}^{\infty} \rho_k(\mathfrak{f}\mathfrak{g})P_{k,k} \tag{6.6}$$

with

$$\rho_k(\mathfrak{f}\mathfrak{g}) = \sum_{m,n=0}^{\infty} \{m,n,k \mid m,n,k\}\,\rho_m(\mathfrak{f})\rho_n(\mathfrak{g}). \tag{6.7}$$

Of course we do not have to express $\rho_k(\mathfrak{f}\mathfrak{g})$ in terms of the SH coefficients, though it is a natural thing to do.

It is similarly natural here to write down the matrix form of equation (6.6), where, as usual, we use the Hermite–Gauss functions for a basis in the domain and codomain. Then

$$\begin{pmatrix} \rho_0(\mathfrak{f}) & 0 & 0 & \dots \\ 0 & \rho_1(\mathfrak{f}) & 0 & \dots \\ 0 & 0 & \rho_3(\mathfrak{f}) & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \odot \begin{pmatrix} \rho_0(\mathfrak{g}) & 0 & 0 & \dots \\ 0 & \rho_1(\mathfrak{g}) & 0 & \dots \\ 0 & 0 & \rho_3(\mathfrak{g}) & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} \rho_0(\mathfrak{f}\mathfrak{g}) & 0 & 0 & \dots \\ 0 & \rho_1(\mathfrak{f}\mathfrak{g}) & 0 & \dots \\ 0 & 0 & \rho_3(\mathfrak{f}\mathfrak{g}) & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \tag{6.8}$$

with the $\rho_k(\mathfrak{f}\mathfrak{g})$ given by equation (6.7). Thus, the diagonal nature of these matrices is preserved under the $\odot$-product.

By way of an example, let us apply the results to positive integral powers of the radius.

Proposition 9. *Let α, β be positive integers and write $\mathfrak{r}^\alpha$ for the distribution taking values $(p^2+q^2)^{\alpha/2}$ and similarly for $\mathfrak{r}^\beta$. Then*

$$\mathbf{\Delta}[\mathfrak{r}^\alpha] \odot \mathbf{\Delta}[\mathfrak{r}^\beta] = \mathbf{\Delta}[\mathfrak{r}^{\alpha+\beta}]. \tag{6.9}$$

The corresponding eigenvalues compose in accordance with

$$\rho_k(\mathfrak{r}^{\alpha+\beta}) = \sum_{m,n=0}^{\infty} \{m,n,k \mid m,n,k\}\, \rho_m(\mathfrak{r}^\alpha)\rho_n(\mathfrak{r}^\beta) \tag{6.10}$$

where

$$\rho_n(\mathfrak{r}^\alpha) = 2^{-\alpha/2} \sum_{j=0}^{\min\{\alpha,n\}} \binom{\alpha}{j} \sqrt{\frac{(n+\alpha-j)!}{(n-j)!}}\, g_{n-j,n-j+\alpha}. \tag{6.11}$$

The g-matrix is characteristic of quantization in polar coordinates, appearing in the integration identity

$$\int_0^\infty [\mathcal{G}(\overline{G_s}\otimes G_t)](r\cos\beta, r\sin\beta) r\,\mathrm{d}r = \frac{1}{2\pi}\sum_{m,n=0}^{\infty} \frac{\mathrm{i}^{m-n}}{2^{m+n}m!n!} g_{m,n} s^m t^n\, \mathrm{e}^{\mathrm{i}(n-m)\beta} \tag{6.12a}$$

holding for all $s, t \in \mathbb{R}$, and with $-\pi < \beta \leqslant \pi$. The g-matrix is rather complicated, and although we do not need its details here, we include these for the sake of completeness: for non-negative integers m, n, let $m \wedge n$ be the minimum of the pair and $m \vee n$ the maximum, and set

$$s(m,n) = \begin{cases} 1/2 & \text{if } m \wedge n \text{ is even} \\ 1 & \text{if } m \wedge n \text{ is odd.} \end{cases} \tag{6.12b}$$

Then

$$g_{m,n} = \sqrt{\frac{m\vee n}{m\wedge n}}\, 2^{|m-n|/2} \frac{\Gamma\left(\frac{1}{2}m\wedge n + s(m,n)\right)}{\Gamma\left(\frac{1}{2}m\vee n + s(m,n)\right)} \tag{6.12c}$$

where Γ is the gamma function. For the analysis of this matrix, see [DHS].

Combining (6.10) and (6.11) results in a quadratic sum law for the g-matrix:

Corollary 10. *For all positive integers j, k and m_1,*

$$\sum_{a=0}^{(j+k)\wedge m_1} \binom{j+k}{a} \sqrt{\frac{(m_1+j+k-a)!}{(m_1-a)!}}\, g_{m_1-a,m_1+j+k-a}$$
$$= \sum_{m_2,m_3=0}^{\infty} \{\boldsymbol{m} \mid \boldsymbol{m}\} \sum_{b=0}^{j\wedge m_2} \sum_{c=0}^{k\wedge m_3} \binom{j}{b}\binom{k}{c} \sqrt{\binom{m_2+j-b}{j}\binom{m_3+k-c}{k}}$$
$$\times\, g_{m_2-b,m_2+j-b}\, g_{m_3-c,m_3+k-c}. \tag{6.13}$$

The significance of this sum law remains to be investigated.

7. Products for angular quantization

The formulae for the quantization of angular distributions are more complex than those for radial distributions, with the g-matrix introduced in the previous section playing an important role. Moreover, the Moyal product of angular distributions is not necessarily an angular distribution, further complicating the subject. However, for the class of angular distributions we consider here, any pair is multipliable, and their product is again an angular distribution. Consequently, the $\odot$-product of angular distributions can be obtained fairly simply.

In writing $p + \mathrm{i}q = r\exp(\mathrm{i}\beta)$, we must specify the function φ that assigns the value β to each point (p, q) in the plane. This requires us to cut the plane, and we choose to do so along the negative abscissa (p-axis). Hence $\varphi\colon \Pi \to [-\pi, \pi)$, with

$$\varphi(p, q) = \beta. \tag{7.1}$$

We note that it is continuous everywhere in the cut plane except across the cut.

The class of angular distributions considered in this paper consists of all tempered distributions of the form

$$\mathfrak{f}_{\mathrm{ang}} = \mathfrak{f} \circ \varphi \tag{7.2}$$

where $\mathfrak{f}\colon \mathbb{T} \to \mathbb{C}$ is continuous and bounded; $\mathbb{T}$ is the unit circle (which we identify with the real interval $[-\pi, \pi)$ when convenient).

Quantization of $\mathfrak{f}_{\mathrm{ang}}$ may be described by specifying its duality pairing with the Hermite–Gauss functions,

$$[\![\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{ang}}]h_n, h_m]\!] = \mathrm{i}^{m-n} g_{m,n} \hat{\mathfrak{f}}_{m-n} \tag{7.3}$$

where $\hat{\mathfrak{f}}_k$ is the kth Fourier component.

Equivalently,

$$\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{ang}}] = \sum_{m,n=0}^{\infty} \mathrm{i}^{m-n} g_{m,n} \hat{\mathfrak{f}}_{m-n} P_{m,n}. \tag{7.4}$$

Proposition 11. *All pairs of angular distributions of the class considered are multipliable, with*

$$(\mathfrak{f}_{\mathrm{ang}})(\mathfrak{h}_{\mathrm{ang}}) = (\mathfrak{f}\mathfrak{h})_{\mathrm{ang}} \tag{7.5}$$

an angular distribution of the same class. Moreover,

$$\boldsymbol{\Delta}[\mathfrak{f}_{\mathrm{ang}}] \odot \boldsymbol{\Delta}[\mathfrak{h}_{\mathrm{ang}}] = \boldsymbol{\Delta}[(\mathfrak{f}\mathfrak{h})_{\mathrm{ang}}]. \tag{7.6}$$

On substituting equation (7.4) into this, we obtain the operator-valued quadratic sum rule

$$\sum_{j,k=0}^{\infty} \mathrm{i}^{j-k} g_{j,k} \widehat{(\mathfrak{f}\mathfrak{h})}_{j-k} P_{j,k} = \sum_{m_1,\ldots,n_3=0}^{\infty} \{\boldsymbol{n} \mid \boldsymbol{m}\}\, \mathrm{i}^{n_1-m_1+n_2-m_2} g_{m_1,n_1} g_{m_2,n_2} \hat{\mathfrak{f}}_{n_1-m_1} \hat{\mathfrak{h}}_{n_2-m_2} P_{m_3,n_3}. \tag{7.7}$$

By choosing specific functions $\mathfrak{f}$ and $\mathfrak{h}$, this last equation implies a number of different sum rules for the g-matrix.

For example, choosing $\widehat{\mathfrak{f}}_k = \delta_{k,s}$ and $\widehat{\mathfrak{h}}_k = \delta_{k,t}$, for $s, t \in \mathbb{Z}$, we find that, for all non-negative $r \in \mathbb{Z}$,

$$g_{r,r+s+t} = \sum_{m_1,m_2=0}^{\infty} \{m_1 + s, m_2 + t, r \mid m_1, m_2, r + s + t\}\, g_{m_1,m_1+s} g_{m_2,m_2+t}. \tag{7.8}$$

This is different in kind from the sum rule (6.13) obtained from radial quantization in that here the sum is over the first two indices; there it is over the last two. We note that if $s < 0$, the sum over m_1 contributes only from $m_1 \geqslant -s$, and similarly for m_2 when $t < 0$. Setting $s = t = 0$ and using the fact that $g_{r,r} = 1$, equation (7.8) implies

$$1 = \sum_{m_1,m_2=0}^{\infty} \{m_1, m_2, r \mid m_1, m_2, r\}. \tag{7.9}$$

8. Conclusions

We have considered the $\odot$-product introduced by Bracken [1], and shown that it is not compatible with either the finite rank operators or the Hilbert–Schmidt operators. Further analysis is necessary to draw conclusions for the other familiar operator classes. To do so will probably require a careful study of the concept of multipliable distributions, including topological considerations. Once one has a topological algebra of multipliable distributions, the $\odot$-product may be seen as a continuous representation into the space $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ of generalized observables.

The principal tool in this paper is the use of the special Hermite functions as a topological basis for the phase space rigged triple. In particular we have provided a solution to the linearization problem for these functions, the necessity of which comes from the pointwise product defining the $\odot$-product. This enables us to express the $\odot$-product of finite rank operators, the corresponding matrices, quantized radial distributions and quantized angle distributions.

Acknowledgments

We would like to thank the anonymous referees for helpful suggestions and comments at the proof stage of this paper.

Appendix A. Linearization for special Hermite functions

As noted above, the introduction of the special Hermite functions as basis elements on phase space will afford us a systematic procedure for considering a number of aspects of the $\odot$-problem, but for this to work we must determine the integral of three special Hermite functions over all of Π. We provide the solution in this section. In fact, it is no more difficult to do so for an arbitrary finite product of d special Hermite functions, so we take the opportunity to do so.

Notation. Until further notice, then, d will be a fixed positive integer with $d \geqslant 3$. Our goal, then, is to determine the *SH coefficients*

$$\{m_1, m_2, \ldots, m_d \,|\, n_1, n_2, \ldots, n_d\} = \frac{1}{2\pi} \int_{\Pi} \prod_{k=1}^{d} \phi_{m_k,n_k}(p, q)\, \mathrm{d}p\, \mathrm{d}q. \tag{A.1}$$

It is convenient to use vector notation for the multiple indices, so we write $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$ and similarly for $\boldsymbol{n}$. This is consistent with their use in the body of the text, cf equation (5.4). Hence we may write the SH coefficients as $\{\boldsymbol{m} \mid \boldsymbol{n}\}$. We will also use the notation $\mu_k = \min\{m_k, n_k\}$, $M_k = \max\{m_k, n_k\}$ and $\delta_k = |m_k - n_k|$ for $k = 1, 2, \ldots, d$. The associated vectors are $\boldsymbol{\mu}$, $\boldsymbol{M}$ and $\boldsymbol{\delta}$. In addition we make use of the l^1-norm: for $\boldsymbol{a} \in \mathbb{R}^d$, $|\boldsymbol{a}| = \sum |a_k|$. The multi-index notation $\boldsymbol{\mu}!$ for $\mu_1! \cdots \mu_d!$ is also useful.

We are going to separate the dth special Hermite function from the first $d-1$ of them in doing the calculation, leading to the appearance of vectors in $\mathbb{R}^{d-1}$ obtained from those of $\mathbb{R}^d$ by striking out the dth component. This will be indicated thus: $\vec{\boldsymbol{a}} = (a_1, \ldots, a_{d-1})$ for $\boldsymbol{a} = (a_1, \ldots, a_d)$.

Thus, we are to find the SH coefficients[11] from the $\mathcal{S}'(\Pi)$-convergent series

$$\prod_{m_k, n_k=0}^{d-1} \phi_{m_k,n_k} = \sum_{m_d,n_d=0}^{\infty} \{\boldsymbol{m} \mid \boldsymbol{n}\}\, \phi_{m_d,n_d}. \tag{A.2}$$

Proposition 12. *The SH coefficients* $\{\boldsymbol{m} \mid \boldsymbol{n}\}$ *for the product of* $d-1$ *special Hermite functions are given by*

$$\{\boldsymbol{m} \mid \boldsymbol{n}\} = (-1)^{|\boldsymbol{\mu}|} \delta_{|\boldsymbol{m}|,|\boldsymbol{n}|} \sqrt{\frac{\boldsymbol{\mu}!}{\boldsymbol{M}!}} \binom{\mu_d + \delta_d}{\delta_d} 2^{d-2} \sum_{k=0}^{|\vec{\boldsymbol{\mu}}|} D_{\vec{\boldsymbol{\mu}}}^{\vec{\boldsymbol{\delta}}}(k) \Gamma(k + 1 + |\boldsymbol{\delta}|/2)$$
$$\times \left(\frac{2}{d}\right)^{k+1+|\boldsymbol{\delta}|/2} {}_2F_1(-\mu_d, k + 1 + |\boldsymbol{\delta}|/2; \delta_d + 1; 2/d) \tag{A.3}$$

where

$$D_{\vec{\boldsymbol{\mu}}}^{\vec{\boldsymbol{\delta}}}(k) = \sum_{|\vec{\boldsymbol{j}}|=k} \Lambda_{\mu_1}^{\delta_1}(j_1) \cdots \Lambda_{\mu_{d-1}}^{\delta_{d-1}}(j_{d-1}) \tag{A.4}$$

with

$$\Lambda_n^a(m) = (-1)^m \binom{n + a}{n - m} \frac{1}{m!}. \tag{A.5}$$

The symbol ${}_2F_1$ *indicates the standard hypergeometric function.*

Proof. The expression for $\phi_{m,n}$ in terms of Laguerre functions, given in equation (3.3c), can be substituted into equation (A.1) for the SH coefficients. This yields

$$\{\boldsymbol{m} \mid \boldsymbol{n}\} = (-1)^{|\boldsymbol{\mu}|} \mathrm{i}^{|\boldsymbol{m}|-|\boldsymbol{n}|} \sqrt{\frac{\boldsymbol{\mu}!}{\boldsymbol{M}!}} 2^{d+\frac{|\boldsymbol{\delta}|}{2}} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(|\boldsymbol{n}|-|\boldsymbol{m}|)\beta} \frac{\mathrm{d}\beta}{2\pi} \int_0^\infty \mathrm{e}^{-\mathrm{d}r^2} r^{|\boldsymbol{\delta}|} \prod_{k=1}^{d} L_{\mu_k}^{\delta_k}(2r^2) r\, \mathrm{d}r$$

so integrating over the angle,

$$= (-1)^{|\boldsymbol{\mu}|} \delta_{|\boldsymbol{m}|,|\boldsymbol{n}|} \sqrt{\frac{\boldsymbol{\mu}!}{\boldsymbol{M}!}} 2^{d+\frac{|\boldsymbol{\delta}|}{2}} \int_0^\infty \mathrm{e}^{-\mathrm{d}r^2} r^{|\boldsymbol{\delta}|} \prod_{k=1}^{d} L_{\mu_k}^{\delta_k}(2r^2) r\, \mathrm{d}r.$$

Now we make the substitution $x = 2r^2$, to get

$$\{\boldsymbol{m} \mid \boldsymbol{n}\} = (-1)^{|\boldsymbol{\mu}|} \delta_{|\boldsymbol{m}|,|\boldsymbol{n}|} \sqrt{\frac{\boldsymbol{\mu}!}{\boldsymbol{M}!}} 2^{d-2} \int_0^\infty \mathrm{e}^{-\frac{d}{2}x} x^{\frac{|\boldsymbol{\delta}|}{2}} \prod_{k=1}^{d} L_{\mu_k}^{\delta_k}(x)\, \mathrm{d}x. \tag{A.6}$$

The series for an associated Laguerre polynomial is (Gradšteĭn and Ryžik [7], 8.970.2)

$$L_n^a(x) = \sum_{m=0}^{n} \Lambda_n^a(m) x^m \tag{A.7}$$

[11] The omission of a d-label should not cause any difficulties.

where $\Lambda_n^a(m)$ is given in equation (A.5). Hence

$$\prod_{r=1}^{d-1} L_{\mu_r}^{\delta_r}(x) = \prod_{r=1}^{d-1}\sum_{j_r=0}^{\mu_r} \Lambda_{\mu_r}^{\delta_r}(j_r)x^{j_r} = \sum_{k=0}^{|\vec{\mu}|} D_{\vec{\mu}}^{\vec{\delta}}(k)x^k \tag{A.8}$$

where $D_v^u(j)$ is defined in equation (A.4).

Substituting equation (A.8) into (A.6),

$$\{\boldsymbol{m} \mid \boldsymbol{n}\} = (-1)^{|\mu|}\delta_{|\boldsymbol{m}|,|\boldsymbol{n}|}\sqrt{\frac{\boldsymbol{\mu}!}{\boldsymbol{M}!}}2^{d-2}\sum_{k=0}^{|\vec{\mu}|} D_{\vec{\mu}}^{\vec{\delta}}(k)\int_0^\infty \mathrm{e}^{-\frac{d}{2}x}x^{k+\frac{|\delta|}{2}}L_{\mu_d}^{\delta_d}(x)\,\mathrm{d}x. \tag{A.9}$$

The integral is known (Gradšteĭn and Ryžik [7] (7.414.7)):

$$\int_0^\infty \mathrm{e}^{-sx}x^\beta L_n^a(x)\,\mathrm{d}x = \binom{n+a}{a}\Gamma(\beta+1)s^{-\beta-1}{}_2F_1\left(-n, \beta+1; a+1; \frac{1}{s}\right). \tag{A.10}$$

The result is now immediate. □

Since the case $d = 3$ is our principal interest, and our solution depends essentially on the $D_{\vec{\mu}}^{\vec{\delta}}(k)$, we give two other forms for it.

Corollary 13. *When $d = 3$,*

$$D_{\vec{\mu}}^{\vec{\delta}}(k) = \frac{(-1)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\binom{\mu_1+\delta_1}{\mu_1-k+j}\binom{\mu_2+\delta_2}{\mu_2-j} \tag{A.11a}$$

$$= \frac{(-1)^k}{k!}\binom{\mu_1+\delta_1}{\mu_1-k}\binom{\mu_2+\delta_2}{\delta_2}{}_3F_2(-k, -k-\delta_1, -\mu_2; \mu_1-k+1, \delta_2+1; -1). \tag{A.11b}$$

As $d = 3$, $\vec{\boldsymbol{\mu}} = (\mu_1, \mu_2)$ and $\vec{\boldsymbol{\delta}} = (\delta_1, \delta_2)$; in the usual Pochhammer notation, ${}_3F_2$ is a Barnes generalized hypergeometric function.

Proof. The first expression for $D_{\vec{\mu}}^{\vec{\delta}}(k)$ follows from the binomial theorem. To obtain the second expression, we proceed as follows. Write the first expression as

$$D_{\vec{\mu}}^{\vec{\delta}}(k) = \frac{(-1)^k}{k!}\sum_{j=0}^{k} c_j$$

so that ($k, \mu_1, \ldots, \delta_2$ are fixed)

$$c_j = \binom{k}{j}\binom{\mu_1+\delta_1}{\mu_1-k+j}\binom{\mu_2+\delta_2}{\delta_2+j}.$$

Now consider the ratio c_{j+1}/c_j,

$$\rho_{j+1,j} = \frac{c_{j+1}}{c_j} = -\frac{(j-k)(j-k-\delta_1)(j-\mu_2)}{(j+\mu_1-k+1)(j+\delta_2+1)(j+1)}$$

which is of the form

$$x\frac{(j+r_1)(j+r_2)(j+r_3)}{(j+s_1)(j+s_2)(j+1)}$$

with

$$r_1 = -k \quad r_2 = -k - \delta_1 \quad r_3 = -\mu_2 \qquad s_1 = \mu_1 - k + 1 \quad s_2 = \delta_2 + 1 \qquad x = -1.$$

Solving for c_j iteratively,

$$c_j = \left(\prod_{i=0}^{j-1} \rho_{i+1,i} \right) c_0.$$

This is the general term of a (3, 2)-hypergeometric polynomial[12],

$$\sum_{j=0}^{k} c_j = c_0 \, {}_3F_2(r_1, r_2, r_3; s_1, s_2; x)$$

with the parameters given above. As

$$c_0 = \binom{\mu_1 + \delta_1}{\mu_1 - k} \binom{\mu_2 + \delta_2}{\delta_2}$$

we obtain the result asserted. □

Corollary 14. *All of the SH coefficients are nonzero. In particular,*

$$\{(0, 0, m) \mid (0, 0, m)\} = 4\left(-\tfrac{1}{3}\right)^{m+1}. \tag{A.12}$$

This implies that for the square of the lowest indexed special Hermite function, the expansion formula is the non-terminating series

$$\phi_{0,0}(p, q)^2 = 4 \sum_{m=0}^{\infty} \left(-\frac{1}{3}\right)^{m+1} \phi_{m,m}(p, q). \tag{A.13}$$

For the same reason, the pointwise product of any pair of special Hermite distributions, $U_{m_1,n_1} U_{m_2,n_2}$, will be a non-terminating series of U_{m_3,n_3}.

We note that the fact that $\phi_{0,0}^2$ is a non-terminating series of the $\phi_{m,m}$ can be obtained by inspection, as $\phi_{0,0}^2 = 4\exp(-2r^2)$ and the exponential in $\phi_{m,m}$ is $\exp(-r^2)$. However, this argument does not provide the necessary series.

References

[1] Bracken A J 2003 Quantum mechanics as an approximation to classical mechanics in Hilbert space *J. Phys. A: Math. Gen.* **36** L329–L35

[2] Weyl H 1927 Quantenmechanik und Gruppentheorie *Z. Phys.* **46** 1–46

[3] Dubin D A, Hennings M A and Smith T B 2000 *Mathematical Aspects of Weyl Quantization and Phase* (Singapore: World Scientific)

[4] Wigner E P 1932 On the quantum correction for thermodynamic equilibrium *Phys. Rev.* **40** 749–59

[5] Moyal J E 1949 Quantum mechanics as a statistical theory *Proc. Cam. Phil. Soc.* **45** 99–124

[6] Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press)

[7] Gradšteĭn I S and Ryžik I S 1963 *Tables of Integrals, Sums, Series and Products* 4th edn (Moscow: Fizmatgiz)

[8] Halmos P R 1982 *A Hilbert Space Problem Book* 2nd edn (New York: Springer)

[9] Bremermann H 1965 *Distributions, Complex Variables and Fourier Transforms* (Reading, MA: Addison-Wesley)

[10] Trèves F 1967 *Topological Vector Spaces, Distributions and Kernels* (New York: Academic)

[11] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)

[12] Thangavelu S 1993 *Lectures on Hermite and Laguerre Expansions* (Princeton, NJ: Princeton University Press)

[13] von Neumann J 1931 Die Eiendeutigkeit der Schrödingerischen Operatoren *Math. Ann.* **104** 570–8

[12] We avoid the two-row notation to save space.

[14] Landsman N P 1988 *Mathematical Topics Between Classical and Quantum Mechanics* (New York: Springer)
[15] Kohn J and Nirenberg L 1965 An algebra of pseudo-differential operators *Commun. Pure Appl. Math.* **18** 269–305
[16] Hörmander L 1990 *The Analysis of Linear Partial Differential Operators I* 2nd edn (Berlin: Springer)
[17] Schwartz L 1950 *Théorie des distributions* vols 1 and 2 (Paris: Hermann)
[18] Gelfand I M and Shilov G E 1964 *Generalized Functions I* (New York: Academic)
[19] Vladimirov V S 1971 *Equations of Mathematical Physics* (New York: Dekker)